Exercise Sheet Solutions #10

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P1. Let $(X, \| \bullet \|_X)$ and $(Y, \| \bullet \|_Y)$ be a normed spaces and $L : X \to Y$ a linear functional. Prove that the following are equivalent:

(a) L is bounded, i.e. that there is $C > 0$ such that for each $x \in X$, $\|L(x)\|_Y \leq C\|x\|_X$,

(b) L is continuous,

(c) L is continuous at 0.

Solution: By linearity of L, **(a)** says that L is Lipschitz which implies **(b)**. Meanwhile **(b)** implies **(c)**, so it is enough to prove that **(c)** implies **(a)**. In fact, as L is continuous at 0, there is $r > 0$ such that $L(B(0, r)) \subseteq B(0, 1)$. Thus, for each $x \in X \setminus \{0\}$ we have that

$$\|L(\frac{r}{2} \cdot \frac{x}{||x||})\| \leq 1,$$

or equivalently

$$\|L(x)\| \leq \frac{2}{r} \cdot ||x||.$$

As the last equality also holds for $x = 0$, calling $C = 2/r > 0$ concludes the implication.

P2. Show that norm vector spaces are topological vector spaces.

Solution: We need to check that the addition map $\varphi : V \times V \to V$, $(u, v) \mapsto \varphi(u, v) = u + v$ is continuous, and that the product by scalar map $\psi : \mathbb{F} \times V \to V$, $(c, v) \to \psi(c, v) = c \cdot v$ is continuous. Indeed, we notice that

$$\|\varphi(u, v)\|_V = \|u + v\|_V \leq \|u\|_V + \|v\|_V = \|(u, v)\|_{V \times V},$$

which implies the continuity of φ by **P1**. On the other hand,

$$\|\psi(c, v)\|_V = \|cv\|_V = |c| \cdot \|v\|.$$

Now, for $(c, v) \in \mathbb{F} \times V$, if $|c - c'| \leq \epsilon$ and $\|v - v'\| \leq \epsilon$ for $\epsilon > 0$ then

$$\begin{aligned}
\|\psi(c, v) - \psi(c', v')\|_V &= \|\psi(c, v) - \psi(c', v) + \psi(c', v) - \psi(c', v')\|_V \\
&= \|\psi(c - c', v) + \psi(c', v' - v)\|_V \leq \|\psi(c - c', v)\| + \|\psi(c', v' - v)\|_V \\
&= |c - c'| \cdot \|v\| + |c'| \cdot \|v' - v\| \\
&\leq \epsilon||v|| + \epsilon|c'| \\
&\leq \epsilon||v|| + \epsilon(|c| + \epsilon),
\end{aligned}$$

which shows the continuity of ψ as well, concluding.

P3. Let $I \subseteq \mathbb{R}$ be an interval, $\varphi : I \to \mathbb{R}$ a convex function. If $t \in \text{Int}(I)$, then $\exists m \in \mathbb{R}$ s.t.

$$\varphi(s) \geq m(s - t) + \varphi(t), \quad \forall s \in I.$$

Solution: Let us prove first that for $w < t < s$ we have that

$$\frac{\varphi(t)-\varphi(w)}{t-w} \leq \frac{\varphi(s)-\varphi(t)}{s-t}.$$

Indeed, we notice that $\lambda = \frac{s-t}{s-w} \in (0,1)$. Thus by convexity:

$$\varphi(t) = \varphi(\lambda w + (1-\lambda)s) \leq \lambda\varphi(w) + (1-\lambda)\varphi(s),$$

or equivalently

$$\lambda(\varphi(t)-\varphi(w)) \leq (1-\lambda)(\varphi(s)-\varphi(t)).$$

simplifying we conclude

$$\frac{\varphi(t)-\varphi(w)}{t-w} \leq \frac{\varphi(s)-\varphi(t)}{s-t}.$$

Now take $s \in \operatorname{int}(I)$. Assume without loss of generality that $s > t$ (the other case is simpler). By convexity, we know that for each $\lambda \in (0,1)$

$$\varphi(\lambda s + (1-\lambda)t) \leq \lambda\varphi(s) + (1-\lambda)\varphi(t).$$

Re-writing the previous inequality we get

$$\frac{\varphi(t+\lambda(s-t))-\varphi(t)}{\lambda} \leq \varphi(s)-\varphi(t).$$

Define

$$m = \lim_{\lambda\to 0} \frac{\varphi(t+\lambda(s-t))-\varphi(t)}{\lambda(s-t)}.$$

If this limit really exists, we would conclude that

$$m(s-t)+\varphi(t) \leq \varphi(s).$$

We notice that the function $u \in (0, s-t) \mapsto \frac{\varphi(t+u)-\varphi(t)}{u}$ is decreasing, and bounded by

$$\frac{\varphi(t)-\varphi(w)}{t-w},$$

for any $w \in I$ with $w < t$. Thus, the aforemetioned limit exists and it is equal to

$$\inf\left\{\frac{\varphi(t+\lambda(s-t))-\varphi(t)}{\lambda(s-t)} \mid \lambda \in (0,1)\right\},$$

concluding.

P4. Let $p, q \in (1,\infty)$ conjugate exponents and $f \in L^p$. Show that

$$\|f\|_p = \sup_{\|g\|_q \leq 1} \left|\int fg d\mu\right|.$$

Solution: We prove the equality by doble inequality. First, we take $q \in L^q$ with $||g||_q \leq 1$. By Holder's inequality

$$\left|\int fg d\mu\right| \leq \int |fg|\, d\mu \leq ||f||_p ||g||_q \leq ||f||_p.$$

Taking supremum we conclude

$$\|f\|_p \geq \sup_{\|g\|_q \leq 1} \left| \int fg d\mu \right|.$$

For the other direction, define

$$g(x) = \begin{cases} \frac{1}{||f||_p^{p/q}} |f|^{p-2}(x) \cdot \overline{f}(x) & \text{if } f(x) \neq 0 \\ 0 & \text{else} \end{cases}.$$

We prove that $g \in L^q$. Indeed, we observe that as $1/p + 1/q = 1$ we have $q(p-1) = p$. This implies

$$\int |g|^q d\mu = \frac{1}{||f||_p^p} \int |f|^{(p-1)q} d\mu = \frac{1}{||f||_p^p} \int |f|^p d\mu = ||f||_p^{p-p} = 1,$$

So, we get that $g \in L^q$ and $||g||_q = 1$. Thus, we have

$$\sup_{\|g'\|_q \leq 1} \left| \int fg' d\mu \right| \geq \left| \int fg d\mu \right| = \frac{1}{||f||_p^{p/q}} \int |f|^p d\mu = ||f||_p^{p-p/q} = ||f||_p,$$

concluding the second inequality and hence the equality.